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# A FITTING THEOREM FOR SIMPLE THEORIES

DANIEL PALACÍN AND FRANK O. WAGNER

ABSTRACT. The Fitting subgroup of a type-definable group in a simple theory is relatively definable and nilpotent. Moreover, the Fitting subgroup of a supersimple hyperdefinable group has a normal hyperdefinable nilpotent subgroup of bounded index, and is itself of bounded index in a hyperdefinable subgroup.

## 1. INTRODUCTION

The *Fitting subgroup*  $F(G)$  of a group  $G$  is the group generated by all normal nilpotent subgroups. Since the product of two normal nilpotent subgroups of class  $c$  and  $c'$  respectively is again a normal nilpotent subgroup of class  $c + c'$ , it is clear that the Fitting subgroup of a finite group is nilpotent. In general, this need not be the case, and some additional finiteness conditions are needed. For groups with the chain condition on centralisers ( $\mathfrak{M}_c$ ), nilpotency of the Fitting subgroup was shown by Bryant [3] for periodic groups, by Poizat and Wagner [9, 12] in the stable case (stability being a model-theoretic condition implying, in particular, that definable groups are  $\mathfrak{M}_c$ ), and generally by Derakhshan and Wagner [4].

In this paper, we shall prove nilpotency of the Fitting subgroup in the model-theoretic context of groups type-definable in simple structures (Theorem 3.9), and virtual nilpotency of the Fitting subgroup for groups hyperdefinable in supersimple structures (Theorem 4.7). Simplicity is a fundamental broadening of stability mentioned above; algebraic examples of simple structures include  $\omega$ -free bounded pseudo-algebraically closed fields, generic (differential) difference fields, or vector spaces over a finite field with a non-degenerate bilinear form.

In this context, a weaker chain condition on centralisers one might call (uniform)  $\widetilde{\mathfrak{M}}_c$  arises naturally [13, Theorem 4.2.12]: The (uniform) chain condition on centralisers up to finite index. More precisely, we shall assume that there are natural numbers  $n, d < \omega$  such that any chain  $(C_G(a_j : j < i) : i < k)$  of centralizers, each of index at least  $d$  in its predecessor, has length  $k < n$ .

Similarly to the approach in [9, 12] we shall also need this chain condition on quotients by relatively definable subgroups, which again follows from simplicity, as the quotients are again type-definable. In this context, in contrast to  $\mathfrak{M}_c$ -groups, the centralizer of an infinite set need not be definable, and the centralizer of a relatively definable set (or even subgroup) need not even be type-definable. Nevertheless, the general theory of simplicity (Fact 2.6) yields the (relative) definability

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of a weaker, still purely group theoretic notion introduced by Haimo [5] under the name of FC-centralizer: For  $H \leq G$  put

$$\tilde{C}_G(H) = \{g \in G : |H : C_H(g)| < \infty\}.$$

However, whereas for subgroups  $H, K \leq G$  trivially  $H \leq C_G(K)$  if and only if  $K \leq C_G(H)$ , no such symmetry has to hold for the FC-centralisers, even if one only asks for inclusion up to finite index. Nevertheless, for type-definable groups in a simple theory, symmetry does hold (Proposition 2.7). Note that Fact 2.6 (and more generally Fact 3.1) and symmetry are the only consequences of simplicity of the ambient theory we will use until we study hyperdefinable groups in section four. In fact, Hempel [6] has recently shown that symmetry of the FC-centraliser holds in general for type-definable groups; adapting our methods she deduces nilpotency of the Fitting subgroup of any group with the uniform  $\widetilde{\mathfrak{M}}_c$ -condition in definable sections.

In the course of the paper, we shall also need that soluble and nilpotent groups are contained (up to finite index) in relatively definable soluble and nilpotent supergroups. In the definable simple case this has been shown by Milliet [7]. We follow his proofs (which adapt ideas from the hyperdefinable case [13]) in the type-definable context, adding some details about the existence of a suitable normal relatively definable abelian/central series.

Some words about definability: A subset of a structure is *definable* if it is the set of realizations of some first-order formula, which may contain parameters, such as

$$\begin{aligned} Z(G) &= \{x : G(x) \wedge \forall y [G(y) \rightarrow xy = yx]\}, \quad \text{or} \\ C_G(a) &= \{x : G(x) \wedge xa = ax\}, \end{aligned}$$

where the group  $G$  is given by a predicate  $G(x)$ , or by a formula abbreviated as  $G(x)$ . A set is definable *relative* to a superset  $X$  if it is the intersection with  $X$  of a definable set; it is *type-definable* if it is given as the intersection of definable sets. Finally,  $X$  is *hyperdefinable* if it is the quotient of a type-definable set in countably many variables by a type-definable equivalence relation. In particular, a type-definable group  $(G, \cdot)$  is a type-definable set  $G$  together with a definable subset of  $G \times G \times G$  which is the graph of the group operation. A subgroup  $H$  of a group  $G$  is relatively definable if the underlying set  $H$  is definable relative to  $G$ . Note that the quotient of a type-definable group by a relatively definable normal subgroup is still type-definable<sup>1</sup>; however the quotient of a type-definable group by a type-definable normal subgroup is only hyperdefinable, which in a way makes hyperdefinable groups a more natural category to work with.

Recall that if  $\kappa$  is an infinite cardinal at least the size of the language, we call a structure  $\kappa$ -saturated if any intersection of less than  $\kappa$  definable sets is non-empty as soon as all finite subintersections are. In the presence of type-definable sets, we shall always assume that the ambient structure is more highly saturated than the size of the intersections used in the type-definitions.

Although we make no assumption on the language used, we can in fact assume that it is reduced to the pure group language  $\{\cdot, {}^{-1}, =\}$  (in addition to the predicate(s) (type-)defining the ambient group). All our definable sets have an easy

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<sup>1</sup>or *type-interpretable*, if we do not consider classes modulo  $\emptyset$ -definable equivalence relations as elements

group-theoretic interpretation (except for Section four where the model theory becomes more involved).

## 2. ALMOST JUST DEFINITIONS

**Definition 2.1.** A subgroup  $H$  of  $G$  is *almost contained* in another subgroup  $K$ , denoted by  $H \lesssim K$ , if  $H \cap K$  has finite index in  $H$ . Then  $H$  and  $K$  are *commensurable*, denoted by  $H \sim K$ , if  $H \lesssim K$  and  $K \lesssim H$ .

Observe that  $\lesssim$  is a transitive relation among subgroups of  $G$ , and that  $\sim$  is an equivalence relation.

**Definition 2.2.** Let  $H$  be a subgroup of a group  $G$ . The *almost normalizer* of  $H$  is defined as

$$\tilde{N}_G(H) = \{g \in G : H \sim H^g\}.$$

Note that if  $H$  and  $K$  are commensurable, then  $\tilde{N}_G(H) = \tilde{N}_G(K)$ . By compactness, if  $G$  is type-definable and  $H$  relatively definable, then  $\tilde{N}_G(H)$  is relatively definable if and only if all  $\tilde{N}_G(H)$ -conjugates of  $H$  are uniformly commensurable with  $H$ ; otherwise  $\tilde{N}_G(H)$  is not even type-definable.

We shall usually work in a context where commensurability is uniform. Then a theorem by Schlichting [11], generalized by Bergmann and Lenstra [2] (see also [13, Theorem 4.2.4] for definability questions), provides an invariant object:

**Fact 2.3.** Let  $\mathcal{H}$  be a family of uniformly commensurable subgroups of a group  $G$ , i.e. the index  $|H : H \cap H^*|$  is finite and bounded independently of  $H, H^* \in \mathcal{H}$ . Then there is a subgroup  $N$  commensurable with any  $H \in \mathcal{H}$ , which is invariant under any automorphism of  $G$  stabilizing  $\mathcal{H}$  setwise. Moreover,  $N$  is a finite extension of a finite intersection of groups in  $\mathcal{H}$ ; if the latter are relatively definable, so is  $N$ .

It follows that if  $H$  is uniformly commensurable with all its  $\tilde{N}_G(H)$ -conjugates, then there is a normal subgroup  $\tilde{H} \trianglelefteq \tilde{N}_G(H)$  commensurable with  $H$ . Clearly, any two choices for  $\tilde{H}$  will be commensurable, and  $N_G(\tilde{H}) = \tilde{N}_G(\tilde{H}) = \tilde{N}_G(H)$ . We call  $\tilde{H}$  a *conjugacy-connected component* of  $H$ . Clearly, any two conjugacy-connected components of  $H$  are commensurable, and have the same normalizer, which is equal to their almost normalizer.

**Definition 2.4.** Let  $K$  and  $H$  be subgroups of a group  $G$  with  $H \lesssim \tilde{N}_G(K)$ , and suppose  $K$  has a conjugacy-connected component  $\tilde{K}$ . The *almost centralizer of  $H$  modulo  $K$*  is given by

$$\tilde{C}_G(H/K) = \{g \in \tilde{N}_G(K) : |H : C_H(g/\tilde{K})| \text{ is finite}\}.$$

For  $n < \omega$  the  $n$ -th iterated almost centralizer of  $H$  modulo  $K$  is defined inductively by  $\tilde{C}_G^0(H/K) = K$ , and if  $\tilde{C}_G^n(H/K)$  has a conjugacy-connected component, then

$$\tilde{C}_G^{n+1}(H/K) = \bigcap_{i \leq n} \tilde{N}_G(\tilde{C}_G^i(H/K)) \cap \tilde{C}_G(H/\tilde{C}_G^n(H/K)).$$

If  $K = \{1\}$  it is omitted. Moreover, we define the *almost centre* of  $G$  as  $\tilde{Z}(G) = \tilde{C}_G(G)$  and for  $n < \omega$  the  $n$ -th iterated almost centre of  $G$  as  $\tilde{Z}_n(G) = \tilde{C}_G^n(G)$ .

Thus  $\tilde{C}_G(H/K) = \tilde{C}_G^1(H/K)$  and  $\tilde{Z}(G) = \tilde{C}_G^1(G)$ . For any subgroup  $L$  we put  $\tilde{C}_L^n(H/K) = \tilde{C}_G^n(H/K) \cap L$  and  $\tilde{N}_L(K) = \tilde{N}_G(K) \cap L$ .

**Remark 2.5.** One easily sees that  $\tilde{C}_G(H/K)$  is an  $H$ -invariant subgroup of  $\tilde{N}_G(K)$ , as are all  $\tilde{C}_G^n(H/K)$  for  $n > 0$ . Moreover, if  $H^* \sim H$  and  $K^* \sim K$ , then  $\tilde{C}_G^n(H/K) = \tilde{C}_G^n(H^*/K^*)$  for all  $n > 0$ .

**Fact 2.6.** *If  $H$  is type-definable and  $K$  a relatively definable subgroup of  $H$  in a simple theory, then commensurability among conjugates is uniform. In particular,  $K$  has a conjugacy-connected component, and both  $\tilde{N}_G(K)$  and  $\tilde{C}_G^n(H/K)$  are relatively definable subgroups of  $G$ .*

*Proof:* This follows immediately from [13, Lemma 4.2.6].  $\square$

In a simple theory, the existence of generic elements yields the following symmetry property, which plays an essential role throughout the paper.

**Proposition 2.7.** *Let  $G$  be type-definable in a simple theory, and  $H$  and  $K$  be type-definable subgroups of  $G$ . The following are equivalent:*

- (1)  $H \lesssim \tilde{C}_G(K)$ .
- (2) *There are independent generic elements  $h \in H$  and  $k \in K$  with  $[h, k] = 1$ .*

*In particular,  $H \lesssim \tilde{C}_G(K)$  if and only if  $K \lesssim \tilde{C}_G(H)$ .*

*Proof:* Suppose  $H \lesssim \tilde{C}_G(K)$ . So there is a generic  $h \in H$  with  $h \in \tilde{C}_G(K)$ . Thus  $C_K(h)$  has finite index in  $K$ , and there is generic  $k \in K$  over  $h$  with  $k \in C_G(h)$ . Then  $h$  and  $k$  are independent, and  $[h, k] = 1$ .

Conversely, suppose  $h \in H$  and  $k \in K$  are independent generics with  $[h, k] = 1$ . As  $k \in C_K(h)$  and  $k$  is generic over  $h$ , the index of  $C_K(h)$  in  $K$  is finite. Thus  $h \in \tilde{C}_G(K)$ ; as  $h$  is generic, we get  $H \lesssim \tilde{C}_G(K)$ .  $\square$

Of course, if  $H, K \leq N_G(N)$  we also have  $H \lesssim \tilde{C}_G(K/N)$  if and only if  $K \lesssim \tilde{C}_G(H/N)$ , by working in the group  $N_G(N)/N$ . Thus symmetry also holds for relative almost centralizers.

We shall finish this section by recalling two group-theoretic facts.

**Fact 2.8** ([8, Theorem 3.1]). *There is a finite bound of the size of conjugacy classes in a group  $G$  if and only if the derived subgroup  $G'$  is finite.*

**Fact 2.9** ([1, 10]). *Let  $H$  and  $N$  be subgroups of  $G$  with  $N$  normalized by  $H$ . If the set of commutators*

$$\{[h, n] : h \in H, n \in N\}$$

*is finite, then the group  $[H, N]$  is finite.*

### 3. NILPOTENCY IN TYPE-DEFINABLE GROUPS IN A SIMPLE THEORY

In this section, we shall generalize the results of Milliet [7] to the relatively definable context. For this we need the following result.

**Fact 3.1** ([13, Theorem 4.2.12]). *Let  $G$  be type-definable in a simple theory, and  $\mathcal{H}$  a family of uniformly relatively definable subgroups. Then there are  $n, d < \omega$  such that any chain  $(\bigcap_{j < i} H_j : i < k)$  of intersections of groups  $H_i \in \mathcal{H}$ , each of index at least  $d$  in its predecessor, has length  $k < n$ .*

**Lemma 3.2.** *Let  $G$  be a type-definable group in a simple theory, and  $H$  a soluble subgroup of  $G$ . Then there is a relatively definable soluble subgroup  $S$  containing  $H$ , and a series of relatively definable  $S$ -invariant subgroups*

$$\{1\} = S_0 < S_1 < \cdots < S_n = S,$$

all normalized by  $N_G(H)$ , such that  $S_i/S_{i+1}$  is abelian for all  $i < n$ . The derived length  $n$  of  $S$  is at most three times the derived length of  $H$ . Moreover,  $S_1$  and  $S_n/S_{n-1}$  are finite.

*Proof:* Suppose first that  $H$  is abelian. By Fact 3.1 there is a finite tuple  $\bar{h} \in H$  and  $d < \omega$  such that for any  $h \in H$

$$|C_G(\bar{h}) : C_G(\bar{h}, h)| < d.$$

Hence the  $N_G(H)$ -conjugates of  $C_G(\bar{h})$  are all commensurable, and  $N_G(H) \leq \tilde{N}_G(C_G(\bar{h})) =: N$ . Now Fact 2.6 yields that  $N$  is relatively definable, and  $C_G(\bar{h})$  has a relatively definable conjugacy-connected component  $C$  normalized by  $N$ . Then

$$H = C_H(\bar{h}) \leq C_G(\bar{h}) \lesssim C.$$

Now, as

$$C \sim C_G(\bar{h}) \sim C_G(\bar{h}, h) = C_{C_G(\bar{h})}(h) \sim C_C(h)$$

for any  $h \in H$ , the relatively definable subgroup  $\tilde{Z}(C)$  of  $G$  contains  $H \cap C$ , whence  $H \lesssim \tilde{Z}(C)$ . By compactness there is a finite bound on the size of conjugacy classes in  $\tilde{Z}(C)$ , so  $\tilde{Z}(C)'$  is finite by Fact 2.8 and hence definable. Put  $S_2 = C_{\tilde{Z}(C)}(\tilde{Z}(C)')$ , a relatively definable characteristic subgroup of  $C$ , which must be normalized by  $N$ . Since  $\tilde{Z}(C)'$  is finite,  $S_2$  has finite index in  $\tilde{Z}(C)$ , so  $H \lesssim S_2$ . If  $S_1 = S_2 \cap \tilde{Z}(C)'$ , then  $S_1$  is finite, abelian (even central in  $S_2$ ) and normalized by  $N$ , and  $S_2/S_1$  is abelian. Put  $S_3 = HS_2$ , a finite extension of  $S_2$  and thus relatively definable. Then  $S_3/S_2$  is abelian as well; moreover, if  $\bar{h}'$  is a system of representatives of  $S_3/S_2$ , then

$$N_G(H) \leq N_1 = \{g \in N : h^g \in S_3 \text{ for all } h \in \bar{h}'\} \leq N_G(S_3).$$

So  $N_1$  is a relatively definable subgroup of  $G$  normalizing  $S_3$  and containing  $N_G(H)$ . Replace  $G$  by  $N_1/S_3$ , we finish by induction.  $\square$

Note that the above proof merely uses the  $\widetilde{\mathfrak{M}}_c$ -condition for  $G$  and for certain relatively definable sections of  $G$ , but not symmetry of the almost centraliser. This is different for nilpotency, where the following lemma is used.

**Lemma 3.3.** *Let  $G$  be a type-definable group in a simple theory. Then there is a characteristic relatively definable subgroup  $G_0$  of finite index and a finite characteristic subgroup  $N \leq Z(G_0)$  such that  $\tilde{Z}(G) \leq C_G(G_0/N)$ .*

*Proof:* As trivially  $\tilde{C}_G(G) \lesssim \tilde{C}_G(G)$ , Proposition 2.7 yields

$$G \lesssim \tilde{C}_G(\tilde{C}_G(G)) = \tilde{C}_G(\tilde{Z}(G)),$$

and so  $\tilde{C}_G(\tilde{Z}(G))$  is a characteristic relatively definable subgroup of finite index in  $G$ . For independent  $g \in \tilde{Z}(G)$  and  $h \in \tilde{C}_G(\tilde{Z}(G))$  we have

$$[g, h] \in \text{acl}(g) \cap \text{acl}(h) = \text{acl}(\emptyset).$$

As every element in  $\tilde{Z}(G)$  is the product of two generic elements  $g, g'$  each of which can be chosen independently of  $h \in \tilde{C}_G(\tilde{Z}(G))$ , and

$$[gg', h] = [g, h][[g, h], g'][g', h] \in \text{acl}(\emptyset),$$

the set of commutators

$$\{[g, h] : g \in \tilde{Z}(G), h \in \tilde{C}_G(\tilde{Z}(G))\}$$

is bounded, whence finite by compactness. By Fact 2.9 the characteristic group  $Z = [\tilde{Z}(G), \tilde{C}_G(\tilde{Z}(G))]$  is finite. We put  $G_0 = \tilde{C}_G(\tilde{Z}(G)) \cap C_G(Z)$  and  $N = Z \cap G_0$ .  $\square$

**Lemma 3.4.** *Let  $G$  be a type-definable group in a simple theory, and  $H$  a nilpotent subgroup of  $G$ . Then there is a relatively definable nilpotent subgroup  $N$  with  $H \lesssim N$ , and a series of relatively definable  $N_G(H)$ -invariant subgroups*

$$\{1\} = N_0 < N_1 < \cdots < N_n = N$$

*such that  $N_{i+1} \leq C_G(N/N_i)$  for all  $i < n$ . The nilpotency class  $n$  of  $N$  is at most two times the nilpotency class of  $H$ .*

*The same conclusion holds if  $H$  is merely FC-nilpotent, i.e.  $H \lesssim \tilde{Z}_k(H)$  for some  $k$ .*

*Proof:* We use induction on the (FC-)nilpotency class of  $H$ . Consider the family  $\mathcal{H} = \{C_G(g) : g \in \tilde{C}_G(H)\}$ . By Fact 3.1 there is a finite intersection  $C$  of groups in  $\mathcal{H}$  such that any further intersection has boundedly finite index. As  $\mathcal{H}$  is clearly  $N_G(H)$ -invariant, we obtain  $\tilde{N}_G(C) \geq N_G(H)$ . Fact 2.6 yields that  $\tilde{N}_G(C)$  is relatively definable, and  $C$  has a relatively definable conjugacy-connected component  $\tilde{C}$ . So  $N_G(\tilde{C}) = \tilde{N}_G(C) \geq N_G(H)$ . As  $H \lesssim C_G(g)$  for any  $C_G(g) \in \mathcal{H}$ , we get

$$H \lesssim C \sim \tilde{C}.$$

By construction

$$\tilde{C}_G(H \cap \tilde{C}) = \tilde{C}_G(H) \leq \tilde{C}_G(C) = \tilde{C}_G(\tilde{C}),$$

whence  $\tilde{Z}(H \cap \tilde{C}) \leq \tilde{Z}(\tilde{C})$ . Now consider the characteristic relatively definable groups  $G_0$  and  $N$  given by Lemma 3.3 applied to  $\tilde{C}$ . Then  $H \lesssim G_0$ . We put  $N_1 = N$  and  $N_2 = \tilde{Z}(\tilde{C}) \cap G_0$ . Note that  $N_1$ ,  $N_2$  and  $G_0$  are all relatively definable and normalized by  $N_G(\tilde{C}) \geq N_G(H)$ , and  $G_0 \cap H$  is normalized by  $N_G(H)$ ; moreover  $N_1 \leq C_G(G_0 \cap H)$  and  $N_2 \leq C_G(G_0 \cap H/N_1)$ . Now  $(H \cap G_0)/N_2$  is a nilpotent subgroup of  $N_G(\tilde{C})/N_2$  of smaller (FC-)nilpotency class, and we finish by induction.  $\square$

**Remark 3.5.** If in addition  $H$  is normalized by  $N$ , then  $NH$  is nilpotent of class at most three times the class of  $H$ ; if  $c$  is the nilpotency class of  $H$  and  $\bar{h} \in H$  is a system of representatives of  $NH/N$ , then

$$\begin{aligned} \{1\} &\leq C_{N_1}(\bar{h}) \leq C_{N_1}^2(\bar{h}) \leq \cdots \leq C_{N_1}^{c+1}(\bar{h}) = N_1 \\ &\leq C_{N_2}(\bar{h}/N_1) \leq C_{N_2}^2(\bar{h}/N_1) \leq \cdots \leq C_{N_2}^{c+1}(\bar{h}) = N_2 \\ &\leq \cdots \leq C_{N_n}^{c+1}(\bar{h}/N_{n-1}) = N_n = N \\ &\leq NZ(H) \leq NZ_2(H) \leq \cdots \leq NZ_c(H) = NH \end{aligned}$$

is a relatively definable central series for  $NH$  normalised by  $N_G(H)$ .

**Lemma 3.6.** *Let  $G$  be a type-definable group in a simple theory and  $\mathcal{H}$  a directed system of nilpotent subgroups. Then  $H = \bigcup \mathcal{H}$  is soluble.*

*Proof:* Let  $(n, d)$  be the bounds given by Fact 3.1 for chains of centralizers, and  $(n', d')$  the bounds for chains of centralizers modulo  $\tilde{Z}(G)$ . The proof is by induction on  $n$ .

Consider  $G_0$  and  $N$  as given by Lemma 3.3. As  $HG_0/G_0$  is finite and nilpotent, we may assume that  $H \leq G_0$ . If  $H \leq \tilde{Z}(G)$  we are done, as  $(\tilde{Z}(G) \cap G_0)/N$  and  $N$  are abelian.

If  $H \not\leq \tilde{Z}(G)$ , then consider some  $H_0 \in \mathcal{H}$  with  $H_0 \not\leq \tilde{Z}(G)$  and we take  $h_0 \in C_{H_0}(H_0/\tilde{Z}(G)) \setminus \tilde{Z}(G)$ ; note that such an element exists as  $H_0/\tilde{Z}(G)$  is nilpotent.

If  $C_H(h_0/\tilde{Z}(G))$  has index greater than  $d'$  in  $H$ , then there is some  $H_1 > H_0$  in  $\mathcal{H}$  such that  $C_{H_1}(h_0/\tilde{Z}(G))$  has index greater than  $d'$  in  $H_1$ . Choose  $h_1 \in C_{H_1}(H_1/\tilde{Z}(G)) \setminus \tilde{Z}(G)$ , and note that  $C_G(h_1, h_0/\tilde{Z}(G))$  has index greater than  $d'$  in  $C_G(h_1/\tilde{Z}(G))$ . If  $C_H(h_1/\tilde{Z}(G))$  has index more than  $d'$  in  $H$ , then we can iterate this process, which must stabilize after at most  $n'$  steps. It follows that there is some  $h \in H \setminus \tilde{Z}(G)$  such that  $C_H(h/\tilde{Z}(G))$  has index at most  $d'$  in  $H$ .

Since  $C_G(h)$  has infinite index in  $G$ , the induction hypothesis for  $n - 1$  yields that  $C_H(h)$  is soluble. Moreover, as  $N$  is central in  $G_0$  the map from  $C_H(h/N)$  to  $N$  given by  $x \mapsto [h, x]$  is a homomorphism with abelian image and kernel  $C_H(h)$ . Thus  $C_H(h/N)/C_H(h)$  is abelian. Similarly, as  $\tilde{Z}(G)$  is centralised by  $H$  modulo  $N$ , the map  $x \mapsto [h, x]N$  is a homomorphism from  $C_H(h/\tilde{Z}(G))$  to  $\tilde{Z}(G)/N$  with abelian image and kernel  $C_H(h/N)$ . Therefore,  $C_H(h/\tilde{Z}(G))$  is soluble. Finally, as  $C_H(h/\tilde{Z}(G))$  contains a normal subgroup  $K$  of  $H$  with  $H/K$  finite, whence nilpotent, we see that  $H$  must be soluble.  $\square$

**Corollary 3.7.** *A locally nilpotent subgroup  $H$  of a type-definable group in a simple theory is soluble.*

*Proof:* The collection of finitely generated subgroups of  $H$  satisfies the hypotheses of Lemma 3.6.  $\square$

**Lemma 3.8.** *Let  $G$  be a type-definable group acting definably on a type-definable abelian group  $A$ , in a simple theory. Suppose that  $H \leq G$  is abelian, and that there are  $\bar{g} = (g_i : i < k)$  in  $H$  and  $m_i < \omega$  for  $i < k$  such that  $(g_i - 1)^{m_i} A$  is finite for all  $i < k$ , and for any  $g \in H$  the index of  $C_A(\bar{g}, g)$  in  $C_A(\bar{g})$  is finite. Put  $m = 1 + \sum_{i < k} (m_i - 1)$ . Then there is a relatively definable supergroup  $\bar{H}$  of  $H$  such that  $\bar{C}_A^m(\bar{H})$  has finite index in  $A$ .*

*Proof:* By [13, Lemma 4.2.6] the group

$$\bar{H} = \{g \in C_G(\bar{g}) : |C_A(\bar{g}) : C_A(\bar{g}, g)| \text{ finite}\}$$

is relatively definable, and it clearly contains  $H$ . By the pigeonhole principle, for any  $m$  indices  $(i_j : j < m) \in k^m$ , there must be at least one  $i < k$  such that  $m_i$  of the indices are equal to  $i$ . As the group ring  $\mathbb{Z}(H)$  is commutative, this implies that

$$(g_{i_0} - 1)(g_{i_1} - 1) \cdots (g_{i_{m-1}} - 1)A$$

is finite. Hence there is a subgroup  $A_0$  of finite index in  $A$  such that

$$(g_{i_0} - 1)(g_{i_1} - 1) \cdots (g_{i_{m-1}} - 1)A_0 = 0$$

for all of the finitely many choices  $(i_j : j < m) \in k^m$ . It follows that for all choices of  $(i_j : 1 \leq j < m) \in k^{m-1}$  we have

$$(g_{i_1} - 1) \cdots (g_{i_{m-1}} - 1)A_0 \leq C_A(g_i : i < k).$$

As  $C_A(h_0, g_i : i < k)$  has finite index in  $C_A(g_i : i < k)$  for all  $h_0 \in \bar{H}$ , the group

$$(h_0 - 1)(g_{i_1} - 1) \cdots (g_{i_{m-1}} - 1)A_0$$

is finite for all choices of  $(i_j : 1 \leq j < m) \in k^{m-1}$ , as is

$$(h_0 - 1)(g_{i_1} - 1) \cdots (g_{i_{m-1}} - 1)A.$$



By the same argument (keeping  $h_0$  fixed) and the fact that  $\bar{H} \leq C_G(\bar{g})$  we see that for any  $h_1$  in  $G$  and all choices of  $(i_j : 2 \leq j < m) \in k^{m-2}$  the group

$$(h_1 - 1)(h_0 - 1)(g_{i_2} - 1) \cdots (g_{i_{m-1}} - 1)A$$

is finite, and inductively that

$$(h_{m-1} - 1) \cdots (h_1 - 1)(h_0 - 1)A$$

is finite for any  $(h_j : j < m)$  in  $\bar{H}$ . It follows that

$$\bar{H} \leq \tilde{C}_{\bar{H}}((h_{m-2} - 1) \cdots (h_0 - 1)A)$$

for all  $(h_j : j < m - 1)$  in  $G$ , whence by symmetry

$$(h_{m-2} - 1) \cdots (h_0 - 1)A \lesssim \tilde{C}_A(\bar{H}).$$

But  $\tilde{C}_A(\bar{H})$  is relatively definable; we may divide out and note that

$$(h_{m-2} - 1) \cdots (h_0 - 1)A / \tilde{C}_A(\bar{H})$$

is finite for all choices of  $(h_j : j < m - 1)$  in  $\bar{H}$ . Hence

$$\bar{H} \leq \tilde{C}_{\bar{H}}((h_{m-3} - 1) \cdots (h_0 - 1)A / \tilde{C}_A(\bar{H}))$$

and by symmetry

$$(h_{m-3} - 1) \cdots (h_0 - 1)A \lesssim \tilde{C}_A(\bar{H} / \tilde{C}_A(\bar{H})) = \tilde{C}_A^2(\bar{H}).$$

Inductively, we see that  $A \lesssim \tilde{C}_A^m(\bar{H})$ .  $\square$

**Theorem 3.9.** *Let  $G$  be a type-definable group in a simple theory. Then the Fitting subgroup  $F(G)$  is nilpotent.*

*Proof:* As  $F(G)$  is soluble by Lemma 3.6, by Lemma 3.2 there is a chain

$$\{1\} = S_0 < S_1 < \cdots < S_d = S$$

of relatively definable normal subgroups of  $G$  such that  $S$  contains  $F(G)$  and all quotients  $S_{i+1}/S_i$  for  $i < d$  are abelian. Since  $F(S_i) = F(G) \cap S_i$ , we may assume by induction on  $d$  that  $F(G)' \leq F(S_{d-1})$  is nilpotent. By Lemma 3.4 and Remark 3.5 there is a relatively definable normal nilpotent group  $N$  containing  $F(G)'$ , and a relatively definable series

$$\{1\} = N_0 < N_1 < \cdots < N_k = N$$

of normal subgroups of  $G$  with  $[N, N_{i+1}] \leq N_i$  for all  $i < k$ .

Fix  $i > 0$ . Any  $g \in F(G)$  is contained in a normal nilpotent subgroup  $H_g$ . Since  $N_i H_g$  is again nilpotent, there is  $m_g < \omega$  such that

$$(g - 1)^{m_g} N_i \leq N_{i-1}.$$

By Fact 3.1 there is a finite tuple  $\bar{g} \in F(G)$  such that for any  $g \in F(G)$  the index  $|C_{N_i}(\bar{g}/N_{i-1}) : C_{N_i}(\bar{g}, g/N_{i-1})|$  is finite. Furthermore, by Lemma 3.8 (applied to  $G/N$  with its abelian subgroup  $F(G)/N$  acting on  $N_i/N_{i-1}$  by conjugation) there is  $m_i < \omega$  and a relatively definable group  $H_i \geq F(G)$  such that  $N_i \lesssim \tilde{C}_G^{m_i}(H_i/N_{i-1})$ . Then, the finite intersection  $\bigcap_i H_i$  is a relatively definable supergroup of  $F(G)$ . By Facts 3.1 and 2.3 there is a relatively definable normal subgroup  $H$  which is a finite extension of a finite intersection of  $G$ -conjugates of  $\bigcap_i H_i$ . Thus  $H \geq F(G)$ , and  $H \lesssim H_i$  for all  $i$ . By Lemma 3.2 applied to the abelian normal subgroup  $F(G)/N$

of  $G/N$ , we may restrict  $H$  and assume that there are relatively definable normal subgroups

$$N \leq Z \leq A \leq H$$

of  $G$  with  $Z/N$  and  $H/A$  finite and  $A/Z$  abelian. Then

$$N_i \lesssim \tilde{C}_G^{m_i}(H_i/N_{i-1}) \leq \tilde{C}_G^{m_i}(H_i \cap H/N_{i-1}) = \tilde{C}_G^{m_i}(H/N_{i-1}),$$

and inductively

$$\begin{aligned} N_i &\lesssim \tilde{C}_G^{m_i}(H/N_{i-1}) \\ &\leq \tilde{C}_G^{m_i}(H/\tilde{C}_G^{m_{i-1}}(H/N_{i-2})) = \tilde{C}_G^{m_i+m_{i-1}}(H/N_{i-2}) \\ &\vdots \\ &\leq \tilde{C}_G^{m_i+m_{i-1}+\dots+m_1}(H). \end{aligned}$$

Thus  $N \lesssim \tilde{C}_N^m(H)$  for  $m = m_1 + m_2 + \dots + m_k$ . Since  $Z \lesssim N$  and  $N \leq A \leq H$  we obtain  $Z \lesssim \tilde{C}_A^m(A)$ , whence  $A = \tilde{Z}^{m+1}(A)$ . So  $A$  is nilpotent-by-finite by Lemma 3.4, as is  $F(G)$ , since  $F(G) \lesssim A$ . If  $F$  is a normal nilpotent subgroup of finite index in  $F(G)$  and  $K$  a normal nilpotent group containing representatives for  $F(G)/F$ , then  $FK$  is a nilpotent, as is  $F(G)$ .  $\square$

#### 4. HYPERDEFINABLE GROUPS

In the previous section we have systematically used the fact that type-definability is preserved under quotients whenever we divide out by a relatively definable subgroup. However, type-definability is not preserved when quotienting by a type-definable subgroup, and in fact such quotients (and even the slightly more general ones defined below) arise naturally from model-theoretic considerations in simplicity theory. We are thus led to the following definition.

**Definition 4.1.** A *hyperdefinable* group is a group whose domain is given by a partial type  $\pi$  modulo a type-definable equivalence relation  $E$ , and whose group law is induced by an  $E$ -invariant type-definable relation on  $\pi^3$ .

Note that a quotient of a hyperdefinable group by a hyperdefinable group is again hyperdefinable. In this context, we have to replace *finite index* by *bounded index*, i.e. the index remains bounded even in a very saturated elementary extension. With this replacement in the definitions of Section 2, almost containment is transitive, commensurability is an equivalence relation, and we still have good definability properties in a simple theory.

**Fact 4.2** ([13, Proposition 4.4.10 and Corollary 4.5.16]). *Let  $K$  and  $H$  be hyperdefinable subgroups of a hyperdefinable group  $G$  in a simple theory, with  $H \lesssim \tilde{N}_G(K)$ . Then:*

- (1)  $\tilde{N}_G(K)$  is hyperdefinable, and a conjugacy-connected component  $\tilde{K}$  exists.
- (2)  $\tilde{C}_G^n(H/K)$  is hyperdefinable for all  $n < \omega$ .

Moreover, the proof of Proposition 2.7 remains valid in the hyperdefinable context.

In contrast to the type-definable case, simplicity does not necessarily yield a finite chain condition on centralizers (even though there is an ordinal  $\alpha$  such that any descending chain of hyperdefinable subgroups having unbounded index in its

predecessor stabilizes, up to bounded index, after  $\alpha$  many steps). In order to adapt the arguments from the previous section we shall make a stronger assumption, supersimplicity. More precisely, we shall assume the following consequence of supersimplicity: There is no infinite descending chain of hyperdefinable subgroups, each of unbounded index in its predecessor. In particular, we obtain a minimal condition on centralizers, up to bounded index.

As a consequence, all proofs of the previous section adapt to this wider context and therefore we obtain the same result, up to bounded index. Note that Remark 3.5 need no longer hold, as a system of representatives for a subgroup of bounded index can now be infinite.

Alternatively, we offer a distinct proof of virtual nilpotency of the Fitting subgroup of a hyperdefinable group of ordinal SU-rank in a simple theory, which in addition provides a bound on the nilpotency class. For the rest of the section, the ambient theory will be simple. We first recall some facts starting with the Lascar inequalities for SU-rank.

**Fact 4.3** ([13, Theorem 5.1.6 (1)]). *If  $H$  and  $K$  are hyperdefinable subgroups of a common hyperdefinable group, then*

$$\text{SU}(K) + \text{SU}(HK/K) \leq \text{SU}(HK) \leq \text{SU}(K) \oplus \text{SU}(HK/K),$$

where  $\oplus$  is the least symmetric increasing function on ordinals satisfying  $f(\alpha, \beta + 1) = f(\alpha, \beta) + 1$ .

**Fact 4.4** ([13, Proposition 5.4.3]). *Let  $G$  be an  $\emptyset$ -hyperdefinable group of rank  $\text{SU}(G) = \omega^\alpha \cdot n + \gamma$  with  $\gamma < \omega^\alpha$ . Then  $G$  has an  $\emptyset$ -hyperdefinable normal subgroup  $H$  of SU-rank  $\omega^\alpha \cdot n$ .*

**Corollary 4.5.** *Let  $G$  be an  $\emptyset$ -hyperdefinable group of rank  $\text{SU}(G) = \omega^{\alpha_1} \cdot n_1 + \dots + \omega^{\alpha_k} \cdot n_k$  with  $\alpha_i > \alpha_{i+1}$  for  $i < k$  and  $n_i > 0$  for  $i \leq k$ . Then there exists a series of  $\emptyset$ -hyperdefinable  $G$ -invariant subgroups*

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_\ell = G$$

with  $\ell \leq n_1 + \dots + n_k$  such that each quotient  $G_{i+1}/G_i$  is unbounded of monomial SU-rank  $\omega^{\beta_i} \cdot m_i$  and its  $\emptyset$ -hyperdefinable  $G$ -invariant subgroups of unbounded index have SU-rank strictly smaller than  $\omega^{\beta_i}$ .

*Proof:* By Fact 4.4 there is an  $\emptyset$ -hyperdefinable normal subgroup  $G_1$  of  $G$  of minimal monomial Lascar rank of the form  $\text{SU}(G_1) = \omega^{\alpha_1} \cdot m$  with positive  $m \leq n_1$ . By minimality,  $G_1$  is as required. If  $\text{SU}(G_1) = \text{SU}(G)$  we are done. Otherwise,  $\text{SU}(G/G_1) < \text{SU}(G)$  by the Lascar inequalities, so we finish by induction on  $\text{SU}(G)$ .  $\square$

Next, we recall the supersimple version of Zilber's Indecomposability Theorem.

**Fact 4.6** ([13, Theorem 5.4.5 and Remark 5.4.7]). *Let  $G$  be an  $\emptyset$ -hyperdefinable group of rank  $\text{SU}(G) < \omega^{\alpha+1}$ . If  $\mathfrak{X}$  is a family of hyperdefinable subsets of  $G$ , then there exists a hyperdefinable subgroup  $K \leq X_1^{\pm 1} \dots X_m^{\pm 1}$  for some  $X_1, \dots, X_m \in \mathfrak{X}$ , such that  $\text{SU}(XK) < \text{SU}(K) + \omega^\alpha$  for all  $X \in \mathfrak{X}$ . Moreover,  $\text{SU}(K) = \omega^\alpha \cdot n$ , and  $K$  is unique up to commensurability. In particular, if  $\mathfrak{X}$  is invariant under all automorphisms we can choose  $K$  hyperdefinable over  $\emptyset$ , and if  $\mathfrak{X}$  is  $G$ -invariant, we can take  $K$  to be normal in  $G$ .*

Finally, we state the hyperdefinable version of our main result in the supersimple case. Recall that the  $\emptyset$ -connected component  $N_\emptyset^0$  of a hyperdefinable group  $N$  is the intersection of all  $\emptyset$ -hyperdefinable subgroups of  $N$  of bounded index in  $N$ .

**Theorem 4.7.** *Let  $G$  be an  $\emptyset$ -hyperdefinable group of rank  $\text{SU}(G) = \omega^{\alpha_1} \cdot n_1 + \dots + \omega^{\alpha_k} \cdot n_k$ . Then  $F(G)$  has bounded index in an  $\emptyset$ -hyperdefinable FC-nilpotent normal subgroup  $N$  of class  $\ell \leq n_1 + \dots + n_k$ . In particular, the hyperdefinable normal group  $N_\emptyset^0$  is nilpotent of class  $2\ell$  and has bounded index in  $F(G)$ .*

*Proof:* By Lemma 4.5 there is a finite series of  $\emptyset$ -hyperdefinable  $G$ -invariant subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_\ell = G$$

with  $\ell \leq n_1 + \dots + n_k$ , such that each quotient  $G_{i+1}/G_i$  is unbounded of monomial Lascar rank  $\omega^{\beta_i} \cdot m_i$ , and its  $\emptyset$ -hyperdefinable  $G$ -invariant subgroups of unbounded index have SU-rank strictly smaller than  $\omega^{\beta_i}$ . Clearly, we may assume that all  $G_i$  are  $\emptyset$ -connected, i.e. have no  $\emptyset$ -hyperdefinable subgroup of bounded index.

Let  $N$  be the intersection  $\bigcap_{i < \ell} \tilde{C}_G(G_{i+1}/G_i)$ , an  $\emptyset$ -hyperdefinable normal subgroup of  $G$ . Note that  $N \leq \tilde{C}_G((G_{i+1} \cap N)/(G_i \cap N))$ . Hence by symmetry we get

$$G_{i+1} \cap N \lesssim \tilde{C}_G(N/(G_i \cap N))$$

for all  $i < \ell$ . Inductively,

$$\begin{aligned} N &= G_\ell \cap N \lesssim \tilde{C}_G(N/(G_{\ell-1} \cap N)) \\ &\leq \tilde{C}_G(N/(\tilde{C}_G(N/(G_{\ell-2} \cap N)))) = \tilde{C}_G^2(N/(G_{\ell-2} \cap N)) \\ &\leq \tilde{C}_G^2(N/\tilde{C}_G(N/(G_{\ell-3} \cap N))) = \tilde{C}_G^3(N/(G_{\ell-3} \cap N)) \\ &\leq \dots \leq \tilde{C}_G^\ell(N/(G_0 \cap N)) = \tilde{C}_G^\ell(N). \end{aligned}$$

It follows that  $N = \tilde{Z}_\ell(N)$  is FC-nilpotent of class at most  $\ell$ . Then  $N_\emptyset^0$  is nilpotent of class  $2\ell$  by [13, Proposition 4.4.10 (3)].

In order to finish, we shall show  $F(G) \leq N$ . Fix  $i \leq \ell$ , and consider the  $\emptyset$ -invariant family  $\mathfrak{X}$  formed by the hyperdefinable sets  $X_a = [a, G_{i+1}]/G_i$  for  $a \in F(G)$ . Suppose, towards a contradiction, that the SU-rank of some of these sets is greater than  $\omega^{\beta_i}$ . By Fact 4.6 applied to  $G_{i+1}/G_i$  we obtain an  $\emptyset$ -hyperdefinable  $G$ -invariant subgroup  $H \leq G_{i+1}/G_i$  of monomial SU-rank which is contained in a finite product of sets  $X_{a_0}^{\pm 1}, \dots, X_{a_m}^{\pm 1}$  from  $\mathfrak{X}$ ; moreover,  $\text{SU}(H) \geq \omega^{\beta_i}$ . Thus  $H$  has bounded index in  $G_{i+1}/G_i$  and must be equal by  $\emptyset$ -connectivity. But every  $a_j G_i$  is contained in a normal nilpotent subgroup of  $G/G_i$  which must also contain  $X_{a_j}$ , so  $K = \langle a_j G_i, X_{a_j} : j \leq m \rangle$  is a nilpotent subgroup of  $G/G_i$ . However,

$$(\dagger) \quad H \leq X_{a_0}^{\pm 1} \dots X_{a_m}^{\pm 1} = [a_0, H]^{\pm 1} \dots [a_m, H]^{\pm 1} \subseteq H$$

and we must have equality, contradicting nilpotency of  $K$ : If  $H$  is in the  $k$ -th element  $\gamma_k(K)$  of the lower central series of  $K$ , then equation  $(\dagger)$  implies that  $H \leq \gamma_{k+1}(K)$ . Thus  $H \leq \bigcap_{k < \omega} \gamma_k(K) = \{1\}$ , a contradiction.

It follows that  $\text{SU}(X_a) < \omega^{\beta_i}$  for all  $X_a \in \mathfrak{X}$ . As  $X_a$  is in bijection with  $G_{i+1}/C_{G_{i+1}}(a/G_i)$ , the Lascar inequalities imply that  $X_a$  is bounded and so  $a \in \tilde{C}_G(G_{i+1}/G_i)$ . Thus  $F(G) \leq N$ , as required.  $\square$

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